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ADDED MASSES OF A CYLINDER INTERSECTING THE INTERFACE OF A TWO-LAYER WEIGHTLESS FLUID OF FINITE DEPTH

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The linear problem of high-frequency oscillations of a horizontal cylinder floating at the interface of a two-layer fluid was solved numerically using the boundary element method. Added masses are calculated for circular and elliptic cylinders.

Key words: *added masses, two-layer fluid, Green's function, method of hybrid finite elements.*

Introduction. When a body moves in an inviscid fluid, the inertial properties of the latter are determined by added masses. For bodies of various shapes moving in a homogeneous fluid, these characteristics have been extensively studied [1]. Density stratification affects added mass values; this effect has been studied most comprehensively for a fluid consisting of two layers of different densities. In those studies, a floating or submerged body was assumed to be within one layer. A recent study of the two-dimensional problem of seakeeping of a body intersecting an interface [2] revealed difficulties that arise in numerical solution of this problem using the method of boundary integral equations (BIE), which is the most commonly used method for determining hydrodynamic loads on bodies.

In the present paper, a numerical solution is obtained using an alternative method of hybrid finite elements (HFE). This is an effective and universal method for solving both two-dimensional and three-dimensional problems of seakeeping theory.

Formulation of the Problem. An ideal incompressible fluid consists of two layers of different densities. Both fluids are assumed to be infinite in the horizontal direction and bounded in the vertical direction by the free surface from above and a flat horizontal bottom from below.

The radiation problem of motions in the fluid initially at rest caused by small high-frequency oscillations of a body. This is equivalent to the assumption that the fluid is weightless; i.e., the accelerations imparted to fluid particles by body oscillations far exceed the acceleration of gravity. This assumption has been widely used in impact theory [1].

The oscillating body is a horizontal cylinder of infinite length; therefore, the problem in question is two-dimensional.

In the absence of the body, the upper fluid layer of density ρ_1 and width H_1 occupies the region $L^{(1)}$ ($|x| < \infty$, $0 < y < H_1$) and the lower fluid layer of density $\rho_2 = \rho_1(1 + \varepsilon)$ and width H_2 occupies the region $L^{(2)}$ ($|x| < \infty$, $-H_2 < y < 0$), where x is the horizontal coordinate and y is the vertical coordinate. The subscripts 1 and 2 correspond to the upper and lower layers. The submerged body intersecting the interface occupies the region $V = V^{(1)} \cup V^{(2)}$. The closed body contour $\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)}$ shares two common points $P_\pm$ with the interface (Fig. 1).

To determine the radiation potentials $\varphi_j^{(s)}(x, y)$ corresponding to horizontal ($j = 1$), vertical ($j = 2$), and rolling ($j = 3$) oscillations, one needs to solve the boundary-value problem (for more detail, see, for example, [2])

$$\Delta\varphi_j^{(s)} = 0, \qquad (x, y) \in L^{(s)} \setminus V^{(s)} \quad (s = 1, 2)$$

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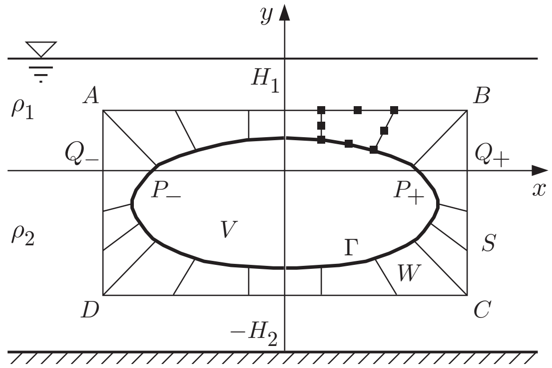

Fig. 1. Scheme of flow and finite elements.

subject to the boundary conditions

$$\varphi_j^{(1)} = 0 \qquad (y = H_1); \tag{1}$$

$$\frac{\partial \varphi_j^{(1)}}{\partial y} = \frac{\partial \varphi_j^{(2)}}{\partial y}, \qquad \rho_1 \varphi_j^{(1)} = \rho_2 \varphi_j^{(2)} \qquad (y = 0); \tag{2}$$

$$\frac{\partial \varphi_j^{(2)}}{\partial y} = 0 \qquad (y = -H_2); \tag{3}$$

$$\frac{\partial \varphi_j^{(s)}}{\partial n} = n_j, \qquad (x, y) \in \Gamma^{(s)}. \tag{4}$$

In Eq. (4), n_1 is the horizontal component of the inner normal to the contour Γ, n_2 is the vertical component, and

$$n_3 = (y - y_0)n_1 - (x - x_0)n_2 \tag{5}$$

(x_0 and y_0 are the coordinates of the point around which rolling oscillations of the body are performed). At a distance from the body, the motion is assumed to decay.

The added-mass coefficients μ_{kj}, characterizing the inertial properties of the fluid, are defined by the formula

$$\mu_{kj} = \sum_{l=1}^{2} \rho_l \int\limits_{\Gamma^{(l)}} \varphi_j^{(l)} n_k \, ds. \tag{6}$$

Numerical Method. The problem formulated above for an arbitrary contour Γ is solved by the HFE method, which has been employed previously to solve the radiation problem for a cylinder completely submerged in the lower layer of a two-layer fluid [3]. In this method, the velocity potentials are represented using finite elements in a narrow region $W = W^{(1)} \cup W^{(2)}$ surrounding the body and using the BIE method in the external region. The region W is bounded from the outside by a rectangular contoured $ABCD$ which contains the specified body (Fig. 1). We denote this rectangular contour by $S = S^{(1)} \cup S^{(2)}$. It intersects the interface at two points $Q_\pm$.

To construct the BIE, one needs to determine Green's function $G^{(s,l)}(x, y; \xi, \eta)$ for the problem considered, where s is the number of the layer containing the point of observation (x, y); the source (ξ, η) is placed in the layer with a number l.

Green's function satisfies the equation

$$\Delta_{x,y} G^{(s,l)} = 2\pi\delta(x - \xi, y - \eta)$$

with the boundary conditions similar to Eqs. (1)–(3) and the condition of damping in the far field and δ is the Dirac delta function.

Green's function can be defined in different ways. In this paper, we use the following representations:

$$G^{(1,1)} = \ln \frac{r}{r_1} + \int\limits_0^\infty \frac{1 + \varepsilon - t_2}{1 + \mathrm{e}^{-2kH_1}} \, [\mathrm{e}^{k(\eta - 2H_1)} - \mathrm{e}^{-k\eta}][\mathrm{e}^{-ky} - \mathrm{e}^{k(y - 2H_1)}] \, D(k, x, \xi) \, dk,$$

$$G^{(2,1)} = 2\int\limits_0^\infty \frac{[\mathrm{e}^{k(\eta-2H_1)} - \mathrm{e}^{-k\eta}][\mathrm{e}^{ky} + \mathrm{e}^{-k(y+2H_2)}]}{(1+\mathrm{e}^{-2kH_1})(1+\mathrm{e}^{-2kH_2})} D(k,x,\xi)\,dk,$$

$$G^{(1,2)} = 2(1+\varepsilon)\int\limits_0^\infty \frac{[\mathrm{e}^{-k(\eta+2H_2)} + \mathrm{e}^{k\eta}][\mathrm{e}^{k(y-2H_1)} - \mathrm{e}^{-ky}]}{(1+\mathrm{e}^{-2kH_1})(1+\mathrm{e}^{-2kH_2})} D(k,x,\xi)\,dk,$$

$$G^{(2,2)} = \ln\frac{r}{r_2} + \int\limits_0^\infty \frac{D(k,x,\xi)\,dk}{1+\mathrm{e}^{-2kH_2}} \Big\{\mathrm{e}^{-k(\eta+H_2)}(\mathrm{e}^{2k\eta}-1)$$

$$\times [(1+\varepsilon+t_1)\mathrm{e}^{-k(y+H_2)} - (1+\varepsilon-t_1)\mathrm{e}^{k(y-H_2)}] - 2t_1\mathrm{e}^{k\eta}[\mathrm{e}^{ky} + \mathrm{e}^{-k(y+2H_2)}]\Big\}.$$

Here

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \qquad r_1 = \sqrt{(x-\xi)^2 + (y+\eta-2H_1)^2},$$

$$r_2 = \sqrt{(x-\xi)^2 + (y+\eta)^2}, \qquad t_1 = \tanh kH_1, \qquad t_2 = \tanh kH_2,$$

$$D(k,x,\xi) = \cos k(x-\xi)/[k(1+\varepsilon+t_1t_2)].$$

For an infinite two-layer fluid ($H_1, H_2 \to \infty$), Green's function is written in simple form [2]:

$$G^{(1,1)} = \ln r + e_1 \ln r_2, \qquad G^{(2,1)} = (1-e_1)\ln r,$$

$$G^{(1,2)} = (1+e_1)\ln r, \qquad G^{(2,2)} = \ln r - e_1 \ln r_2, \qquad e_1 = \varepsilon/(2+\varepsilon).$$

The system of BIE is derived as described in [4]. For the problem considered, this system is written as

$$\rho_m \varphi_j^{(m)}(\zeta) = \frac{1}{\alpha}\sum_{l=1}^{2} \rho_l \int\limits_{S^{(l)}} \Big[\varphi_j^{(l)}(z)\,\frac{\partial G^{(l,m)}(z,\zeta)}{\partial n_z} - G^{(l,m)}(z,\zeta)\,\frac{\partial \varphi_j^{(l)}}{\partial n}\Big]\,ds \quad (m=1,2),$$

$$z = x+iy, \qquad \zeta = \xi + i\eta. \tag{7}$$

During counterclockwise motion along the contour W, $\alpha = 3\pi/2$ for the vertices A, B, C, and D and $\alpha = \pi$ for all other points.

When the HFE method is used, the region W is covered with quadrangular elements (Fig. 1). In this case, the segments Q_+B and Q_-A are divided into NY1 equal parts, the segments CQ_+ and DQ_- are divided into NY2 parts, and the segments AB and DC, into NX parts. The segments P_+Q_+ and Q_-P_- are the element boundaries.

Using Green's theorem, we obtain

$$\sum_{l=1}^{2}\Big[\iint\limits_{W^{(l)}} \nabla\varphi_j^{(l)}\,\nabla\psi\,dx\,dy - \int\limits_{S^{(l)}} \frac{\partial\varphi_j^{(l)}}{\partial n}\,\psi\,ds\Big] = \sum_{l=1}^{2}\int\limits_{\Gamma^{(l)}} \frac{\partial\varphi_j^{(l)}}{\partial n}\,\psi\,ds, \tag{8}$$

where $\psi(x,y)$ is an arbitrary weight function. For each element, we introduce eight-point quadratic isoparametric shape functions N_k ($k = 1,\ldots,8$). The derivatives $\partial\varphi_j^{(l)}/\partial n$ on the right side of Eq. (8) are known from boundary condition (4), and on the left side, they are determined from the system of BIE (7) for S. The relationship between the vectors $\mathbf{\Phi}_j$ with the components $\varphi_j^{(l)}$ and the vectors $\mathbf{\Psi}_j$ with the components $\partial\varphi_j^{(l)}/\partial n$ at the nodes of the contour S is found from Eq. (7) in matrix form using the analog of the shape functions N_k for the one-dimensional case:

$$A\mathbf{\Phi}_j = B\mathbf{\Psi}_j.$$

Here A and B are quadratic matrices of dimensionality $2M$ [$M = 2(\mathrm{NY1}+\mathrm{NY2}+\mathrm{NX})$ is the number of elements in the region W]. From this relation, we obtain the relations

$$\mathbf{\Psi}_j = C\mathbf{\Phi}_j, \qquad C = B^{-1}A$$

and use them in the corresponding discrete form (8) replacing ψ by the functions N_k.

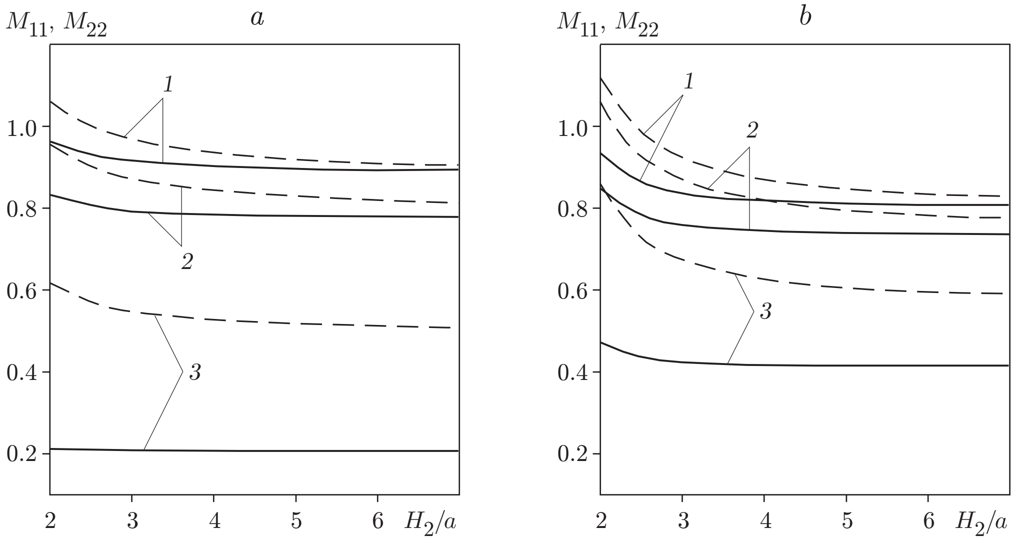

Fig. 2. Added masses M_{11} (solid curves) and M_{22} (dashed curves) versus the depth of the lower layer for a circular cylinder for $h = 0$ and $H_1/a = 2$ (a) and $h/a = 0.5$ and $H_1/a = 1$ (b): $\varepsilon = 0$ (1), 0.3 (2), and ∞ (3).

Thus, for each $j = 1, 2, 3$, we must solve a system of linear equations of order $5M$ to determine the values of $\varphi_j^{(l)}$ for all nodal points. An advantage of the HFE method is that integration over the body contour, whose shape can be rather complicated, is substituted in (7) by analytically integrating over a rectangular contour. The hydrodynamic load is determined after calculating the integrals in (6).

Calculation Results. Numerical calculations were performed for the elliptic contour

$$x^2/a^2 + (y + h)^2/b^2 = 1,$$

where a and b are the major and minor semiaxes of the ellipse, respectively, and h is the depth of submergence of its center under the interface. In all these calculations, the width of the rectangular contour S is $2a + 0.4b$ and its height is $2.4b$.

It is known that for a contour symmetric about the vertical axis y, only the added-mass coefficients μ_{jj} ($j = 1, 2, 3$) and μ_{13} are nonzero. The dimensionless values of these coefficients were normalized as follows:

$$(M_{11}, M_{22}) = (\mu_{11}, \mu_{22})/(\pi\rho_2 b^2), \qquad M_{33} = \mu_{33}/(\pi\rho_2 b^4), \qquad M_{13} = \mu_{13}/(\pi\rho_2 b^3).$$

Figure 2 shows curves of the coefficients M_{11} and M_{22} for a circular cylinder ($a = b$) versus the lower-layer depth H_2. For $h = 0$, the numerical solution was performed for the values NX = 6, NY1 = 3, and NY2 = 3 (number of elements $M = 24$), and for $h = 0.5b$, we used NX = 7, NY1 = 2, and NY2 = 5 ($M = 28$). In the absence of stratification ($\varepsilon = 0$), the problem reduces to determining the added masses of a completely submerged cylinder. This problem was studied in detail, for example, in [1]. It is known that as $H_2 \to \infty$, $M_{11} = M_{22}$. As $\varepsilon \to \infty$, the initial problem is equivalent to determining the added masses of a body floating on the free surface of a homogeneous fluid. The case of vertical oscillations of a semicircle floating on the surface of a fluid of finite depth is considered in [5]. A comparison of the tabular values of M_{22} given in [5] with the results obtained by the proposed method showed that they agree with an accuracy of up to 0.5%.

The limiting values of M_{11} and M_{22} as $H_1, H_2 \to \infty$ for the examined depths of submergence of a circular cylinder and density jumps ε are given in [2].

Calculation results for an elliptic contour ($a/b = 2$) are shown in Fig. 3 for $h = 0$ and $H_1 = 2b$ and in Fig. 4 for $h = 0.5b$ and $H_1 = b$. Rolling oscillations are performed about the geometrical center of the ellipse, i.e., $x_0 = 0$ and $y_0 = -h$ in (5).

In the numerical calculations, we set NX = 12, NY1 = 3, and NY2 = 3 ($M = 36$) for $h = 0$ and NX = 14, NY1 = 2, and NY2 = 5 ($M = 42$) for $h = 0.5b$. Table 1 lists the limiting values of the added-mass coefficients as $H_1, H_2 \to \infty$ for this elliptic contour.

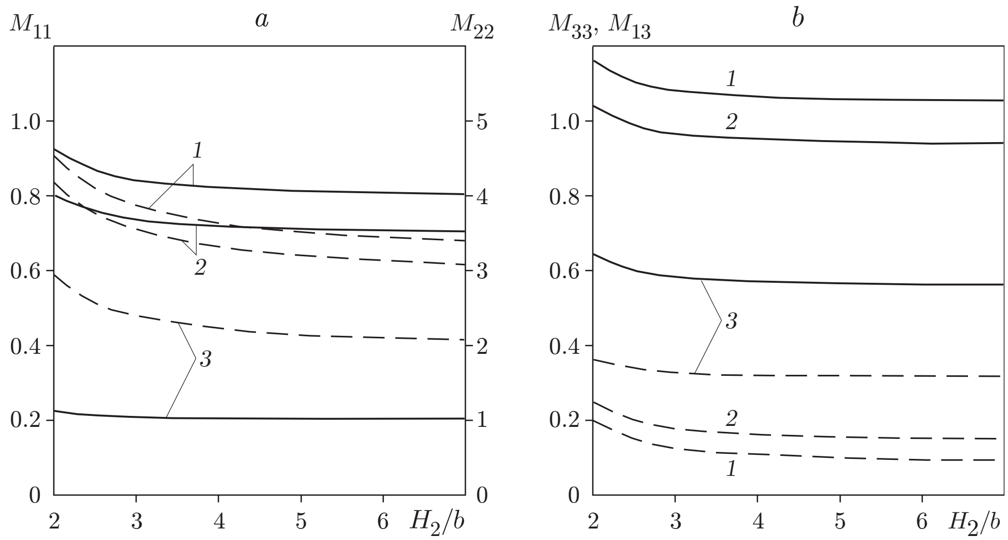

Fig. 3. Added masses versus lower-layer depth for an elliptic cylinder for $a/b = 2$, $h = 0$, and $H_1/b = 2$ and $\varepsilon = 0$ (1), 0.3 (2), and ∞ (3): solid curves refer to M_{11} and M_{33} and dashed curves refer to M_{22} and M_{13}.

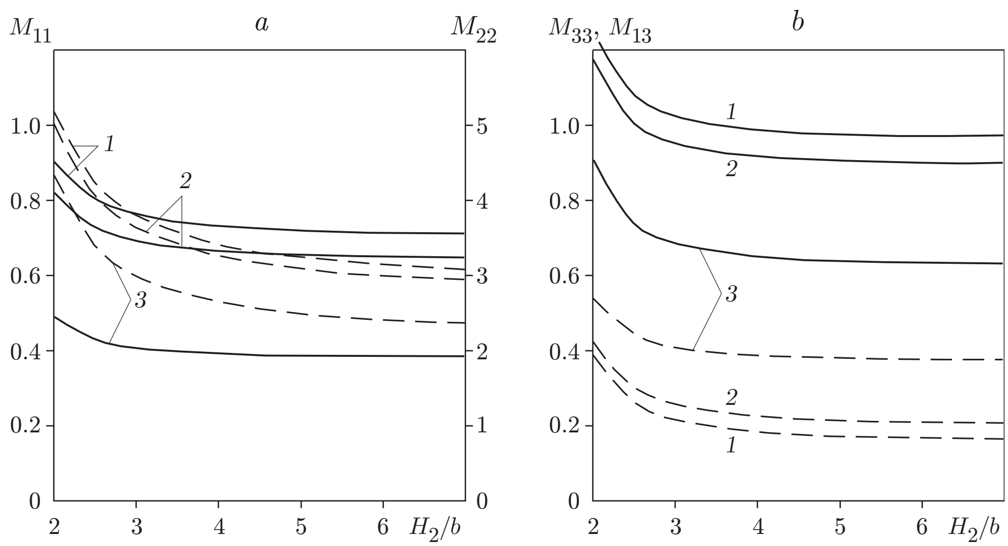

Fig. 4. Added masses versus lower-layer depth for an elliptic cylinder for $a/b = 2$, $h/b = 0.5$, and $H_1/b = 1$ (notation the same as in Fig. 3).

TABLE 1

ε	$h/b = 0.5$				$h = 0$			
	M_{11}	M_{22}	M_{33}	M_{13}	M_{11}	M_{22}	M_{33}	M_{13}
∞	0.386	2.236	0.631	0.373	0.203	2.001	0.562	0.318
0.3	0.898	3.617	1.021	0.072	0.875	3.537	0.994	0.073
0	1.000	3.999	1.123	0	1.000	3.999	1.123	0

It is known that for a semiellipse floating on the free surface of a homogeneous fluid of finite depth [6], we have

$$\mu_{11} = 2\rho_2 b^2/\pi, \quad \mu_{22} = \pi\rho_2 a^2/2, \quad \mu_{33} = \pi\rho_2(a^2 - b^2)^2/16, \quad \mu_{13} = \rho_2 b(a^2 - b^2)/3.$$

Then, for the specified elongation of the ellipse ($a/b = 2$), we obtain the following added-mass coefficients:

$$M_{11} \approx 0.203, \qquad M_{22} = 2, \qquad M_{33} \approx 0.563, \qquad M_{13} \approx 0.318.$$

These coefficients differ by less than 0.2% from the corresponding coefficients listed in Table 1 for $h = 0$ and $\varepsilon = \infty$. The same error takes place for $\varepsilon = 0$, which corresponds to ellipse oscillations in an infinite homogeneous fluid. In this case, only the following diagonal coefficients are nonzero:

$$\mu_{11} = \pi\rho_1 b^2, \qquad \mu_{22} = \pi\rho_1 a^2, \qquad \mu_{33} = \pi\rho_1(a^2 - b^2)^2/8.$$

Hence, for this ellipse, $M_{11} = 1$, $M_{22} = 4$, and $M_{33} = 1.125$.

The results presented above show that the effect of the finite depths of the layers is significant. Korotkin [1] noted that for an oscillating body floating on the free surface of a homogeneous fluid of finite depth H, the bottom has almost no effect if $H \geqslant 4T$ (T is the draft of the body). In the two-layer fluid considered, the effect of the finite dimensions of the layers is insignificant if a similar condition is satisfied in both the upper and lower layers. A decrease in the upper-layer density, i. e. an increase in ε, leads to a decrease in the diagonal added mass coefficients M_{jj} ($j = 1, 2, 3$) and an increase in the coefficient M_{13}.

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REFERENCES

1. A. I. Korotkin, *Added Masses of Ships: Handbook* [in Russian], Sudostroenie, Leningrad (1986).
2. O. V. Motygin and I. V. Sturova, "Wave motions in a two-layer fluid generated by small oscillations of a cylinder intersecting the interface," *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza*, No. 4, 105–119 (2002).
3. I. V. Sturova, "Two-dimensional problem of the hydrodynamic rolling of a submerged body at rest in a two-layer fluid," *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza*, No. 3, 144–155 (1994).
4. A. V. Klimenko, "The two-dimensional Neumann–Kelvin problem for an interface-intersecting body in a two-layer fluid," in: *Day on Diffraction*: Proc. of the Int. Seminar (St. Petersburg, June 1–3, 1999), St. Petersburg (1999), pp. 103–112.
5. K. J. Bai, "The added mass of two-dimensional cylinders heaving in water of finite depth," *J. Fluid Mech.*, **81**, Part 1, 85–105 (1977).
6. M. D. Khaskind, *Hydrodynamic Theory of Ship Motion* [in Russian], Nauka, Moscow (1973).